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ASYMPTOTIC APPROXIMATION OF BIDIMENSIONAL MODIFIED GAMMA OPERATORS

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Abstract

We present an asymptotic approximation result concerning the order of approximation of bivariate functions by means of the bidimensional operators of the Gamma type operators, which was first given in [1] for one variable functions and later was given for two variables function in [2].

1. Introduction

The aim of this present article is to give an asymptotic approximation of bivariate functions, which have second partial derivatives and mixt partial derivatives by means of bidimensional operators of the Gamma type operators.

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Izgi and Büyükyazıcı [1] gave the following operators:

$$A_n(f;\ x) = \int_0^\infty K_n(x,\ t)f(t)dt,$$

where $K_n(x,\ t) = \frac{(2n+3)!\, x^{n+3}}{n!\,(n+2)!} \times \frac{t^n}{(x+t)^{2n+4}}$. For the process of these operators, see [9], [10], and [1], respectively.

These operators preserve x^2, which has better approximation than which does not preserve x^2, as it was shown by King in [8] for the Bernstein polynomials.

It was studied rate of convergence of these operators for functions with derivatives of bounded variation in [5] and also it was studied the rate of pointwise convergence of these operators on the set of functions with bounded variation in [6]. Also, it was studied direct local and global approximation results for these operators in [7]. The present author studied these operators for Voronovskaya type asymptotic approximation in [4], also studied these operators for L^p-integrable functions in [3], and studied these operators for bivariate functions in the weighted spaces with the following operators in [2]:

$$A_{n,m}(f(u,\ v);\ x,\ y) = \int_0^\infty\int_0^\infty K_n(x,\ u)K_m(y,\ v)f(u,\ v)dudv. \tag{1}$$

Lemma 1 ([2]).

$$A_{n,m}(1;\ x,\ y) = 1; \tag{2}$$

$$A_{n,m}(u+v;\ x,\ y) = x + y - \frac{x}{(n+2)} - \frac{y}{(m+2)}; \tag{3}$$

$$A_{n,m}(u^2 + v^2;\ x,\ y) = x^2 + y^2; \tag{4}$$

$$A_{n,m}(u^3 + v^3;\ x) = x^3 + y^3 + \frac{3}{n}x^3 + \frac{3}{m}y^3; \tag{5}$$

$$A_{n,m}(t^4 + u^4;\ x) = x^4 + y^4 + \frac{4(2n+3)}{n(n-1)}x^4 + \frac{4(2m+3)}{m(m-1)}y^4,$$

$$n > 1,\ m > 1; \tag{6}$$

$$A_{n,m}(\{u + v\} - \{x + y\};\ x,\ y) = -\frac{x}{(n+2)} - \frac{y}{(m+2)}; \tag{7}$$

$$A_{n,m}(\{(u - x)^2 + (v - y)^2\};\ x,\ y) = \frac{2x^2}{(n+2)} + \frac{2y^2}{(m+2)}; \tag{8}$$

$$A_{n,m}(\{(u - x)^4 + (v - y)^4\};\ x,\ y) = \frac{12(n+4)}{(n+2)n(n-1)}x^4 + \frac{12(m+4)}{(m+2)m(m-1)}y^4,$$

$$n > 1,\ m > 1. \tag{9}$$

2. Main Result

Let D_{AB} shows $(0, A] \times (0, B]$, where $A > 0$ and $B > 0$. Let $C^r(D_{AB})$ shows the continuous functions on D_{AB}, which have r-th $(r = 0, 1, 2, \ldots)$ partial derivatives and mixt partial derivatives. And define

$$e_{ij} := e_{ij}(x,\ y) = x^i y^j,$$

$$E_{ij} := E_{ij}(u - x, v - y) = (u - x)^i (v - y)^j, \quad (i,\ j = 0, 1, 2, \ldots).$$

In this study, we aim to give an asymptotic approximation for the (1), where $m = n$. Mean for the following operators:

$$A_n(f(u, v);\ x,\ y) := A_{n,n}(f(u, v);\ x,\ y)$$

$$= \int_0^\infty \int_0^\infty K_n(x,\ u) K_n(y,\ v) f(u,\ v)\, du dv. \tag{10}$$

Theorem 1. *If* $f \in C^2(D_{AB})$, *then we have*

$$\lim_{n\to\infty}(n+2)[A_n(f(u, v); x, y) - f(x, y)] = -x\frac{\partial}{\partial x}f(x, y) - y\frac{\partial}{\partial y}f(x, y)$$

$$+ x^2\frac{\partial^2}{\partial x^2}f(x, y) + y^2\frac{\partial^2}{\partial y^2}f(x, y).$$

Proof. Using the corresponding Taylor series, then we have

$$f(u, v)$$

$$= f(x, y) + \left[(u-x)\frac{\partial}{\partial x}f(x, y) + (v-y)\frac{\partial}{\partial y}f(x, y)\right]$$

$$+ \frac{1}{2!}\left[(u-x)^2\frac{\partial^2}{\partial x^2}f(x, y) + (u-x)(v-y)\left(\frac{\partial^2}{\partial x\partial y}f(x, y) + \frac{\partial^2}{\partial y\partial x}f(x, y)\right)\right.$$

$$\left. + (v-y)^2\frac{\partial^2}{\partial y^2}f(x, y)\right] + (u-x)^2\varepsilon_1(u-x) + (v-y)^2\varepsilon_2(v-y)$$

$$+ (u-x)(v-y)\varepsilon_3(u-x, v-y),$$

where the mappings $\varepsilon_i(i = 1, 2, 3)$ are bounded and $\lim_{h\to0}\varepsilon_1(h) = \lim_{h\to0}\varepsilon_2(h) = 0$ and $\lim_{\substack{h_1\to0\\ h_2\to0}}\varepsilon_3(h_1, h_2) = 0.$

Now, we have the following inequality because of (10):

$$A_n(f(u, v); x, y) - f(x, y)$$

$$= A_n(E_{10}; x, y)\frac{\partial}{\partial x}f(x, y) + A_n(E_{01}; x, y)\frac{\partial}{\partial y}f(x, y)$$

$$+ \frac{1}{2}A_n(E_{20}; x, y)\frac{\partial^2}{\partial x^2}f(x, y) + \frac{1}{2}A_n(E_{02}; x, y)\frac{\partial^2}{\partial y^2}f(x, y)$$

$$+ \frac{1}{2}A_n(E_{11}; x, y)\left\{\frac{\partial^2 f(x, y)}{\partial x\partial y} + \frac{\partial^2 f(x, y)}{\partial y\partial x}\right\} + R_n(f; x, y),$$

where

$$R_n(f;\, x,\, y) = A_n(\varepsilon_1(u-x)E_{20};\, x,\, y) + A_n(\varepsilon_2(v-y)E_{02};\, x,\, y)$$
$$+\, A_n(\varepsilon_3(u-x,\, v-y)E_{10}E_{01};\, x,\, y).$$

Using Lemma 1, we get

$$A_n(f(u,\, v);\, x,\, y) - f(x,\, y) = -\frac{x}{(n+2)}\frac{\partial}{\partial x}f(x,\, y) - \frac{y}{(n+2)}\frac{\partial}{\partial y}f(x,\, y)$$
$$+\frac{x^2}{(n+2)}\frac{\partial^2}{\partial x^2}f(x,\, y) + \frac{y^2}{(n+2)}\frac{\partial^2}{\partial y^2}f(x,\, y)$$
$$+\frac{1}{2}\frac{xy}{(n+2)^2}\left\{\frac{\partial^2 f(x,y)}{\partial x\partial y}+\frac{\partial^2 f(x,y)}{\partial y\partial x}\right\}+R_n(f;x,y);$$

and also we have

$$(n+2)\{A_n(f(u,\, v);\, x,\, y) - f(x,\, y)\}$$
$$= -x\frac{\partial}{\partial x}f(x,\, y) - y\frac{\partial}{\partial y}f(x,\, y) + x^2\frac{\partial^2}{\partial x^2}f(x,\, y)$$
$$+\, y^2\frac{\partial^2}{\partial y^2}f(x,\, y) + \frac{1}{2}\frac{xy}{(n+2)}\left\{\frac{\partial^2 f(x,\, y)}{\partial x\partial y}+\frac{\partial^2 f(x,\, y)}{\partial y\partial x}\right\}$$
$$+\, (n+2)R_n(f;\, x,\, y).$$

It is obvious that

$\lim_{n\to\infty}\frac{1}{2}\frac{xy}{(n+2)}\left\{\frac{\partial^2 f(x,\, y)}{\partial x\partial y}+\frac{\partial^2 f(x,\, y)}{\partial y\partial x}\right\} = 0$ for $(x,\, y)\in D_{AB}$. Now, we need to prove $\lim_{n\to\infty}(n+2)R_n(f;\, x,\, y) = 0$ to complete the proof.

Define

$$N_1 = \{(x,\, u)\in N : |u-x| < \delta\}, \quad N_2 = \{(x,\, u)\in N : |u-x| \geq \delta\},$$

$M_1 = \{(v,\, y)\in M : |v-y| < \delta\}$, and finally, $M_2 = \{(v,\, y)\in M : |v-y| \geq \delta\}$, where $N = (0,\, A]\times(0,\, \infty)$ and $M = (0,\, \infty)\times(0,\, B]$.

Now get $\varepsilon > 0$. Since $\lim_{h\to 0} \varepsilon_i(h) = 0$, there is $\delta_i > 0\,(i = 1, 2)$ such that for each $|h| < \delta_i$, $|\varepsilon_i(h)| < \varepsilon$ hold. Besides, since $\lim_{\substack{h_1\to 0\\ h_2\to 0}} \varepsilon_3(h_1, h_2) = 0$, then there is $\delta_3 > 0$ such that for each $\sqrt{h^2 + k^2} < \delta_3$, $|\varepsilon_3(h, k)| < \varepsilon$ hold. Choose $\delta = \max\{\delta_1, \delta_2, \delta_3\}$, then we have $|\varepsilon_i(h)| < \varepsilon(i = 1, 2)$, for each $|h| < \delta$ and $|\varepsilon_3(h, k)| < \varepsilon$, where $\sqrt{h^2 + k^2} < \sqrt{2}\delta$. Then if $(\alpha_1, \beta_1) \in N_2$, $(\alpha_2, \beta_2) \in M_2$, then, we have

$$|\varepsilon_1(\beta_1 - \alpha_1)| \le \sup_{(\alpha_1,\beta_1)\in N_2} |\varepsilon_1(\beta_1 - \alpha_1)| = \|\varepsilon_1\|_\infty,$$

$$|\varepsilon_2(\beta_2 - \alpha_2)| \le \sup_{(\alpha_2,\beta_2)\in M_2} |\varepsilon_2(\beta_2 - \alpha_2)| = \|\varepsilon_2\|_\infty,$$

and

$$|\varepsilon_3(\beta_1 - \alpha_1, \beta_2 - \alpha_2)| \le \sup_{\substack{(\alpha_1,\beta_1)\in N_2\\ (\alpha_2,\beta_2)\in M_2}} |\varepsilon_3| = \|\varepsilon_3\|_\infty.$$

$|h| \ge \delta$ implies $\left(\frac{h}{\delta}\right)^2 \ge 1$ and that implies $h^2 \le \frac{h^4}{\delta^2}$ and also $|h| \ge \delta$, $|k| \ge \delta$ implies $\frac{|h|}{\delta} \ge 1$ and $\frac{|k|}{\delta} \ge 1$ and that implies $|hk| \le \frac{h^2k^2}{\delta^2}$. Then, we have

$$\begin{aligned}
|A_n(\varepsilon_1(u - x)E_{20};\, x,\, y)| &= |A_n(\varepsilon_1(u - x)E_{20};\, x,\, y)|_{|u-x|<\delta} \\
&\quad + |A_n(\varepsilon_1(u - x)E_{20};\, x,\, y)|_{|u-x|\ge\delta} \\
&< \varepsilon \frac{2x^2}{(n + 2)} + \frac{\|\varepsilon_1\|_\infty}{\delta^2} \left|A_n((u - x)^4;\, x,\, y)\right| \\
&< \varepsilon \frac{2A^2}{(n + 2)} + \frac{\|\varepsilon_1\|_\infty}{\delta^2} \frac{12(n + 4)}{(n + 2)n(n - 1)} a^4,
\end{aligned}$$

for $(x,\ y) \in D_{AB}$. Similarly,

$$|A_n(\varepsilon_2(v-y)E_{02};\ x,\ y)| < \varepsilon \frac{2B^2}{(n+2)} + \frac{\|\varepsilon_2\|_\infty}{\delta^2} \frac{12(n+4)}{(n+2)n(n-1)} b^4,$$

and

$$|A_n(\varepsilon_3(u-x,\ v-y)E_{10}E_{01};\ x,\ y)| < |A_n(\varepsilon_3(u-x,\ v-y)E_{10}E_{01};\ x,\ y)|_{\substack{|u-x|<\delta \\ |v-y|<\delta}}$$

$$+ |A_n(\varepsilon_3(u-x,\ v-y)E_{10}E_{01};\ x,\ y)|_{\substack{|u-x|\geq\delta \\ |v-y|\geq\delta}}$$

$$< \varepsilon \frac{xy}{(n+2)^2} + \frac{\|\varepsilon_3\|_\infty}{\delta^2} \frac{2x^2}{(n+2)} \frac{2y^2}{(n+2)}$$

$$< \varepsilon \frac{AB}{(n+2)^2} + \frac{\|\varepsilon_3\|_\infty}{\delta^2} \frac{2A^2}{(n+2)} \frac{2B^2}{(n+2)}.$$

Now, we can estimate $(n+2)R_n(f;\ x,\ y)$ for $n \to \infty$.

$$|(n+2)R_n(f;\ x,\ y)| < 2A^2\varepsilon + \frac{\|\varepsilon_1\|_\infty}{\delta^2} \frac{12(n+4)}{(n+2)n(n-1)} A^4$$

$$+ 2B^2\varepsilon + \frac{\|\varepsilon_2\|_\infty}{\delta^2} \frac{12(n+4)}{(n+2)n(n-1)} B^4$$

$$+ \frac{AB}{(n+2)}\varepsilon + \frac{\|\varepsilon_3\|_\infty}{\delta^2} \frac{4A^2B^2}{(n+2)}.$$

That means

$$\lim_{n\to\infty} (n+2)R_n(f;\ x,\ y) = 0.$$

Then the proof is completed.

Note that; if we examine (1) instead of (10) in the Theorem 1, we will get the following inequality:

$$\lim_{n\to\infty} \min\{(n+2),\ (m+2)\}\left[A_{n,m}(f(u,v);\ x,\ y) - f(x,\ y)\right]$$

$$\leq -x\frac{\partial}{\partial x}f(x,\ y) - y\frac{\partial}{\partial y}f(x,\ y) + x^2\frac{\partial^2}{\partial x^2}f(x,\ y)$$

$$+\ y^2\frac{\partial^2}{\partial y^2}f(x,\ y).$$

Example 1. Consider $f(x,\ y) = (xy)^4$. Then $\dfrac{\partial f(x,\ y)}{\partial x} = 4x^3y^4$,

$$\frac{\partial f(x,\ y)}{\partial y} = 4x^4y^3, \quad \frac{\partial^2 f(x,\ y)}{\partial x^2} = 12x^2y^4, \quad \frac{\partial^2 f(x,\ y)}{\partial y^2} = 12x^4y^2.$$

$$-x\frac{\partial}{\partial x}f(x,\ y) - y\frac{\partial}{\partial y}f(x,\ y) + x^2\frac{\partial^2}{\partial x^2}f(x,\ y) + y^2\frac{\partial^2}{\partial y^2}f(x,\ y)$$

$$= -4x^4y^4 - 4x^4y^4 + 12x^4y^4 + 12x^4y^4 = 16x^4y^4;$$

$$A_n((uv)^4;\ x,\ y) = \left(x^4 + \frac{4(2n+3)}{n(n-1)}x^4\right)\left(y^4 + \frac{4(2n+3)}{n(n-1)}y^4\right)$$

$$= (xy)^4\left(1 + \frac{4(2n+3)}{n(n-1)}\right)^2;$$

$$(n+2)\left|A_n((uv)^4;\ x,\ y) - (xy)^4\right|$$

$$= (xy)^4(n+2)\left[\left(1 + \frac{4(2n+3)}{n(n-1)}\right)^2 - 1\right]$$

$$= (xy)^4\left[\frac{8(2n+3)(n+2)}{n(n-1)} + (n+2)\frac{4(2n+3)^2}{n^2(n-1)^2}\right].$$

Take $\lim\limits_{n\to\infty}$ on both side in the last equality, we get

$$\lim_{n\to\infty}(n+2)\left|A_n((uv)^4;\ x,\ y) - (xy)^4\right| = 16x^4y^4.$$

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